FUNCTORIAL DESINGULARIZATION OVER Q: BOUNDARIES AND THE EMBEDDED CASE

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ABSTRACT. An ordered boundary on a scheme is an ordered set of Cartier divisors. We study various operations on boundaries, including transforms under blow ups. Furthermore, we introduce B-schemes as schemes with boundaries, study their basic properties and interpret them as log-schemes whose stalks of monoids are free. Then we establish functorial desingularization of noetherian quasi-excellent B-schemes of characteristic zero, and deduce functorial embedded desingularization of quasi-excellent schemes of characteristic zero. Finally, a standard simple argument is used to extend these results to other categories, and this includes, in particular, (equivariant) embedded desingularization of the following objects of characteristic zero: qe algebraic stacks, qe schemes, qe formal schemes, complex and non-archimedean analytic spaces.

1. INTRODUCTION

Very often one divides various desingularization problems into two large types: non-embedded desingularization and embedded desingularization. A typical example of a problem of the first type is to associate to a scheme X a blow up sequence $f : X' \dashrightarrow X$ with regular X' and such that f is an isomorphism over the regular locus of X. A typical (but rather crude) example of a problem of the second type is to associate to a regular ambient scheme X with a divisor $Z \hookrightarrow X$ a blow up sequence $f : X' \dashrightarrow X$ which blows up only regular subschemes in the preimage of Z (so X' is regular and f is an isomorphism over $X \setminus Z$) and $f^{-1}(Z)$ is an snc divisor. Although there are much finer versions of embedded desingularization, it seems that the one we have mentioned covers most of the applications of embedded desingularization.

This work is a direct continuation of [Tem2], where functorial non-embedded desingularization of varieties of characteristic zero was used to prove an analogous result for all quasi-excellent schemes of characteristic zero. Our aim is to apply/extend the technique developed in [Tem2] to functorial embedded desingularization of qe schemes of characteristic zero. In particular, we establish the above version of embedded desingularization for such schemes (which is the most general class of schemes over $\mathbf{Q}$ for which the problem can be solved in view of [EGA, IV$_2$, §7.9]). Actually, we solve a finer problem as formulated in Theorem 1.1.6, though when compared with the varieties, this is still far from the strongest known version (we do not achieve principalization and our method does not choose centers that

have simple normal crossings with exceptional divisors). Then, as a simple corollary one obtains the same embedded desingularization result for (formal) qe stacks, and complex/non-archimedean analytic spaces. Also, one obtains equivariant embedded desingularization of all these objects with respect to an action of a regular group.

1.1. **Main results.** Now, we are going to formulate our main results. We try to make the formulations as self-contained as possible, though certain referencing to the terminology introduced later is still involved. Mainly, one has to use the notions of principal and complete transforms of the boundary in order to formulate the sharpest results. Our results are concerned with desingularization of the following objects: a divisor on a regular scheme, a generically reduced scheme with a boundary, a scheme embedded into a regular scheme with an snc boundary. Although, using the non-embedded desingularization from [Tem2] all three results can be easily obtained one from another, we decided to formulate them all since each of them has its own flavor. The first and the third cases are the classical embedded desingularization problems. The second formulation is important for us because the entire paper is written in the language of B-schemes (i.e. schemes with boundaries). The reasons for choosing this language will be discussed in §1.2.1 and §A.1.

1.1.1. *Desingularization of divisors.* In many applications of embedded desingularization one wants to resolve a divisor E (or a function) on a regular ambient variety X by finding a modification $f : X' \to X$ with regular X' and such that the reduction of $E \times_X X'$ is snc (i.e. strictly normal crossings). One can also achieve that f modifies only the non-regular locus of E, but the problem of preserving the entire locus where E is (strictly) normal crossings is more delicate, and, probably, is not the "correct problem" (see §A.1.3). It turns out that the problem that makes much more sense is to preserve the locus where E is snc and the splitting to components is fixed in some sense, and a standard way to formulate this is to consider a *divisorial boundary* $E = \{E_1, \ldots, E_n\}$ where each E_i is a divisor.

Remark 1.1.1. (i) We make our life easier by considering only *ordered* boundaries (otherwise only the new boundary would be ordered accordingly to the history, and this would require to use a heavier terminology, as one does in [CJS]). In particular, we will use the order whenever this shortens our argument. Our desingularization procedure depends on the order of the components and is only compatible with regular morphisms that preserve the order.

(ii) At least in the case of varieties one can functorially desingularize unordered boundaries so that the entire snc locus is preserved, see Remark A.1.1(iii). So, almost surely all our results have "unordered" analogs (where all initial boundaries are unordered).

We refer to §2.1 for the definitions of snc and strictly monomial boundaries, and to §2.2 for the definition of complete and principal transforms of a boundary under a blow up sequence. For reader's convenience basic properties of the transforms are collected in Lemma 2.2.5.

Theorem 1.1.2. *For any quasi-excellent noetherian regular scheme X of characteristic zero and a divisorial boundary E on X there exists a blow up sequence $f = \mathcal{F}_{\mathrm{div}}(X, E) : X' \dashrightarrow X$ such that*

(i) the centers of f are regular (and so X' is regular) and contained in the preimage of the non-snc locus of E,

(ii) the complete transform $f^\circ(E)$ is snc; in particular, the strict transform $f^!(E)$ is snc and the total transform $E \times_X X'$ is strictly monomial,

(iii) $\mathcal{F}_{\mathrm{div}}$ is functorial in exact regular morphisms; that is, given a regular morphism $g : Y \to X$ and $D = E \times_X Y$ the blow up sequence $\mathcal{F}_{\mathrm{div}}(Y, D)$ is obtained from $g^*(\mathcal{F}_{\mathrm{div}}(X, E))$ by omitting all empty blow ups.

Remark 1.1.3. (i) Unlike the known algorithms for varieties, we do not achieve that the centers are transversal to the new boundary.

(ii) Using this algorithm one can obviously construct an algorithm that outputs empty $f^\triangleright(E)$, but necessarily blows up the entire preimage of E (just blow up the components of $f^\triangleright(E)$ accordingly to their order).

1.1.2. *Desingularization of B-schemes.* One does not have to require that X is regular in the above Theorem. Also, it is convenient to allow non-divisorial boundaries $B = \{B_1, \ldots, B_n\}$, where each component B_i is only a locally principal closed subscheme. A pair (X, B) will be called a *B-scheme*, and we will work in the framework of B-schemes in this paper. In particular, the version of the main theorem we will deal with in the paper is given below. In this Theorem, a B-scheme (X', B') is said to be *semi-regular* if X' is regular and B' becomes snc after removing connected components of X' from the boundary components $B_i' \in B'$.

Theorem 1.1.4. *For any quasi-excellent noetherian generically reduced B-scheme (X, B) of characteristic zero there exists a blow up sequence $f = \mathcal{F}(X, B) : X' \dashrightarrow X$ such that*

(i) (X', B') is semi-regular where $B' = f^\circ(B)$,

(ii) each center of f is regular and is disjoint from the preimage of the semi-regular locus of (X, B),

(iii) $\mathcal{F}$ is functorial in exact regular morphisms; that is, given a regular morphism $g : Y \to X$ with $D = B \times_X Y$, the blow up sequence $\mathcal{F}(Y, D)$ is obtained from $g^*(\mathcal{F}(X, B))$ by omitting all empty blow ups.

Remark 1.1.5. (i) Given the non-embedded desingularization Theorem [Tem2, 1.2.1], Theorems 1.1.2 and 1.1.4 are equivalent because one can construct $\mathcal{F}(X, B)$ as the composition of the non-embedded desingularization $g = \mathcal{F}_{\mathrm{non-emb}}(X) : X' \dashrightarrow X$ with $\mathcal{F}_{\mathrm{div}}(X', g^\circ(B)) \dashrightarrow X'$ (where one simply ignores the components of X' in $g^\circ(B)$).

(ii) We require that X is generically reduced because this is our assumption in the non-embedded desingularization in [Tem2]. Using a finer non-embedded desingularization, as mentioned in [Tem2, Rem. 1.2.2(ii)], one would be able to refine Theorem 1.1.4 in a similar way.

1.1.3. *Embedded desingularization.* Here is the strongest version of embedded desingularization which is achieved by our method so far, and we will see in §3.5.2 that it follows easily from Theorem 1.1.4. The main weakness of this variant is that it does not provide strong principalization in the sense of §1.1.4. In addition, the center of the i-th blow up $X_i \to X_{i-1}$ does not have to be transversal to the boundary E_{i-1} (which is the complete transform of $E_0 = E$), and hence the intermediate boundaries E_i can be singular (though the final boundary E_n is snc).

Theorem 1.1.6. *For any quasi-excellent regular noetherian scheme X of characteristic zero with an snc boundary E and a closed subscheme $Z \hookrightarrow X$ there exists a blow up sequence $f = \mathcal{F}_{\mathrm{emb}}(X, E, Z) : X' \dashrightarrow X$ such that*

(i) X' is regular, $E' = f^{\circ}(E)$ is snc and $Z' = f^{!}(Z)$ is regular and has simple normal crossings with E',

(ii) each center of f is regular and for any point x of its image in X either Z is not regular at x or Z has not simple normal crossings with E at x,

(iii) $\mathcal{F}_{\mathrm{emb}}$ is functorial in exact regular morphisms; that is, given a regular morphism $g : Y \to X$ with $D = E \times_X Y$ and $T = Z \times_X Y$, the blow up sequence $\mathcal{F}_{\mathrm{emb}}(Y, D, T)$ is obtained from $g^*(\mathcal{F}_{\mathrm{emb}}(X, E, Z))$ by omitting all empty blow ups.

1.1.4. *Principalization.* In the case of varieties, one can strengthen the above Theorem by adding the condition that the principal transform $f^{\triangleright}(Z)$ equals to the strict transform $f^{!}(Z)$. In particular, this implies that $Z \times_X X' = Z' + E_Z$ where E_Z is a strictly monomial exceptional divisor, and so after adding a blow up along Z' one obtains a strong *principalization* of Z by a blow up sequence $X'' \dashrightarrow X$ (i.e. $Z \times_X X''$ is an exceptional divisor, which is strictly monomial because E'' is snc). This condition cannot be achieved by use of our results on B-schemes, so we do not obtain strong principalization for general qe schemes over $\mathbf{Q}$. Perhaps, the latter could be proved by our general method, but that would require to also run the entire procedure of §§3.1-3.4 in the context of principalization. Therefore we prefer to only establish a weaker principalization, which obviously follows from our other results but suffices for many applications.

Theorem 1.1.7. *For any quasi-excellent noetherian scheme X of characteristic zero with a closed subscheme $Z \hookrightarrow X$ there exists a $(Z \cup X_{\mathrm{sing}})$-supported blow up sequence $\mathcal{F}_{\mathrm{princ}}(X, Z) : X' \dashrightarrow X$ such that X' is regular, $Z \times_X X'$ is strictly monomial and $\mathcal{F}_{\mathrm{princ}}$ is functorial in exact regular morphisms.*

Proof. Let $f : X' \to X$ be the blow up along Z and $Z' = Z \times_X X'$. Then $B' = \{Z'\}$ is a boundary on X' and we can consider the desingularization $g = \mathcal{F}(X', B') : (X'', B'') \dashrightarrow (X', B')$ of the B-scheme (X', B'). Note that g is Z-supported because the bad locus of (X', B') sits over Z. Also, $Z \times_X X''$ is a sum of the components of the snc boundary B'' and hence is strictly monomial. Thus, we can define $\mathcal{F}_{\mathrm{princ}}(X, B)$ as the composition $X'' \dashrightarrow X$. $\square$

1.1.5. *Other categories.* Using the same argument as in [Tem2, §5] one can use the main desingularization theorems for noetherian qe schemes to prove their analogs for other (quasi-compact or not) geometric objects of characteristic zero. Also, it follows from the functoriality that the obtained desingularizations are equivariant.

Theorem 1.1.8. *(i) The functors $\mathcal{F}$, $\mathcal{F}_{\mathrm{div}}$, $\mathcal{F}_{\mathrm{emb}}$ and $\mathcal{F}_{\mathrm{princ}}$ induce analogous functors for quasi-compact (formal) qe stacks and complex/non-archimedean analytic spaces of characteristic zero.*

(ii) All these functors extend to not quasi-compact objects at cost of replacing blow up sequences with blow up hyper-sequences (or simply with a desingularization morphism $X' \to X$ without a blow up sequence structure).

(iii) All these desingularizations are equivariant with respect to any action of a regular group.

1.2. **Overview.** Now, let us discuss briefly the structure of the paper.

1.2.1. *B-schemes.* We devote §2 to defining and basic study of B-schemes, B-blow ups, desingularization of B-schemes, etc. In particular, we define boundaries and their transforms under blow up sequences. Such objects naturally arise in the desingularization theory, though they were formally introduced only very recently in [CJS]. Actually, if one simply wants to restrict an snc boundary E on a regular scheme X onto a closed subscheme $Z \hookrightarrow X$ without excluding degenerate cases, then the restriction $E|_Z$ is a (non-divisorial) boundary. We make one further step with respect to [CJS] by linking schemes and boundaries into a single object – a B-scheme. A partial justification for such terminology is that the B-schemes admit a very nice interpretation as log-schemes (X, M) such that all stalks $\overline{M}_x$ are free monoids $\mathbf{N}^{k(x)}$. Although, we do not use this interpretation in the paper, it might appear to be important in further research.

It should be noted that when working on an earlier version of this paper I observed that the usual non-embedded desingularization implicitly desingularizes a B-scheme $(Z, \emptyset)$ (which is stronger than desingularization of Z), and a similar result is true for non-empty initial boundaries (see §A.1.2). This provides a strong motivation to view the procedure as desingularization of a B-scheme via embedding it into a regular B-scheme, and I wondered whether B-schemes (or even general log-schemes) provide the natural general framework for desingularization theory. The decision to adopt the language of B-schemes was obtained when I saw [CJS] and its technique of working with general boundaries. In particular, the notions of principal and complete transforms are borrowed from [CJS]. There are other arguments in favor of working with B-schemes which we discuss in the appendix.

1.2.2. *Desingularization functors.* In §3 we prove our main results on desingularization of B-schemes. This is done in four steps worked out in §§3.1–3.4. In §3.1 we establish the case of B-varieties by constructing a functor $\mathcal{F}_{\mathrm{BVar}}$. In large, we simply apply the non-embedded desingularization to X and then apply the embedded desingularization to the boundary. However, one must be slightly more careful in order not to destroy the entire snc locus of B, and for this we add an intermediate step in which we separate the old boundary from the singular locus. Then, we extend in §3.2 the functor $\mathcal{F}_{\mathrm{BVar}}$ to a functor $\widehat{\mathcal{F}}_{\mathrm{BVar}}$ on formal B-varieties with small boundary and singular locus. A similar step is the heart of [Tem2], but, fortunately, the argument from [Tem2] extends verbatim to our more general situation. In §3.3 we desingularize a B-scheme (X, B) with a fixed divisor $Z \hookrightarrow X$ which contains the bad locus $(X, B)_{\mathrm{sreg}}$ and is a disjoint union of varieties. The first and main step is to separate the old boundary from the bad locus, and it is done in the same way as in the case of varieties. After that the formal completion of (X, B) along Z can be desingularized by $\widehat{\mathcal{F}}_{\mathrm{BVar}}$. Moreover, that desingularization blows up only open ideals and hence algebraizes to a desingularization of (X, B). Finally, in §3.4 we construct a desingularization $\mathcal{F}(X, B)$ of general qe B-schemes (X, B) of characteristic zero. This is based on the desingularization of B-schemes with small bad locus and is done by induction on codimension similarly to the proof of [Tem2, Th. 1.2.1] in [Tem2, §4.3].

1.2.3. *The appendix.* One could write this paper without using the notion of B-schemes and even without using not snc boundaries (clearly, this would require to formulate the main results in another but equivalent way). Nevertheless, the language of B-schemes seems to be very natural for our task and we discuss the

reasons for this in the appendix. The appendix is not used in the paper, but it can be worth for looking through if the reader suspects that our terminology is artificial (at least we try to convince the reader in the opposite). In particular, we explain in §A.1.3 why a naive boundary, which is a divisor, would not work as fine as our notion, and also correct a mistake in [Tem1] which was caused by a confusion between these two. Also, we formulate the strongest conjecture about desingularization of B-schemes in §A.1.8.

1.2.4. *Conventiones.* We keep all conventiones from [Tem2, §2], including the convention that a blow up sequence remembers the centers of all blow ups. In addition, all (formal) schemes are assumed to be locally noetherian. By a *component* of a scheme X we mean a disjoint union of few connected components of X. Assume that X is a scheme with a closed subscheme Z. Then by $|Z|$ we denote the support of Z (which is the underlying closed set) and by $\mathcal{I}_Z \subset \mathcal{O}_X$ we denote its ideal (and so $Z = \mathbf{Spec}(\mathcal{O}_X/\mathcal{I}_Z)$). We say that Z is *locally principal* (resp. a *Cartier divisor*) if $\mathcal{I}_Z$ is locally principal (resp. invertible). If $D \hookrightarrow X$ is a Cartier divisor then for any $n \in \mathbf{N}$ we define $Z + nD$ to be the closed subscheme corresponding to $\mathcal{I}_Z \mathcal{I}_D^n$. Note also that the fractional ideal $\mathcal{I}_Z \mathcal{I}_D^{-n}$ is an ideal if and only if $nD \hookrightarrow Z$, and in this case we denote the corresponding subscheme as $Z - nD$. Given a morphism $f : X' \to X$ it will be convenient to use the notation $f^*(Z) := Z \times_X X'$ for the pullback of Z and when f is an immersion we will often call to $f^*(Z)$ the *restriction* of Z onto X' and will denote it as $Z|_{X'}$. Also, in this case for any morphism $g : Y \to X$ (e.g. a blow up) we will write $g|_{X'} = g \times_X X'$.

Contents

2. Boundaries and desingularization

2.1. **Schemes with boundaries.**

2.1.1. *Boundary.* In this paper we will only work with ordered boundaries so by a *boundary* on a scheme X we mean a tuple $B = \{B_i\}_{i \in I}$ indexed by a finite ordered set I in which each $B_i \hookrightarrow X$ is a locally principal subscheme. It can freely happen that $B_i = B_j$ for $i \neq j$. Each B_i is called a *component* of B or a *boundary component.* We will ignore empty components and will be only interested in the equivalence class of the ordered index set I. So, any boundary can be uniquely represented in its *reduced form* as $B = \{B_1, \ldots, B_n\}$, where all B_i's are non-empty. We say that two boundaries on X are *equal* if their reduced forms are equal. It is convenient to consider not only boundaries in the reduced form because the latter does not have to be preserved by operations (e.g. some components can vanish when restricting on a closed subscheme $X' \hookrightarrow X$). We define the *ordered disjoint union* of boundaries as $B \sqcup_{ord} B' = \{B_1, \ldots, B_n, B'_1, \ldots, B'_{n'}\}$. The *support* of B is the closed subset $|B| = \cup_{i \in I} |B_i|$ of X, and we also define a finer *schematical support* of B as $[B] = \sum_{i \in I} B_i$ (it is well defined even for non-divisorial boundaries).

Remark 2.1.1. (i) An analogous definition of boundaries is given in [CJS], where one prefers the non-ordered variant of the definition.

(ii) We do not require that the elements that locally define B_i's are not zero divisors, so B_i's do not have to be Cartier divisors. This is convenient because we can then restrict a boundary onto any closed subscheme.

2.1.2. *The stratification induced by B.* For each point $x \in X$ by $I(x)$ (resp. $\widetilde{I}(X)$) we denote the set of all $i \in I$ such that $x \in B_i$ (resp. and B_i is not the whole X in a neighborhood of x, i.e. the element defining B_i does not vanish in $\mathcal{O}_{X,x}$). Also, for any subset $J = \{j_1, \ldots, j_m\} \subset I$ we set $\overline{B}_J := B_{j_1} \times_X \cdots \times_X B_{j_m}$ (so, $\overline{B}_\emptyset = X$) and define the J-th stratum B_J of B as the open subscheme of $\overline{B}_J$ obtained by removing each $\overline{B}_{J'}$ with $J \subsetneq J'$. In particular, $x \in B_{I(x)}$.

Remark 2.1.2. Although, we will not need the following observation in this paper, it can give an alternative point of view on the nature of B. It is equivalent to give an unordered boundary B (indexed by an unordered finite set) or to give a log-structure M on X such that for each point $x \in X$ the monoid $\overline{M}_x$ is free. Actually, under this correspondence one has that $\overline{M}_x \widetilde{\rightarrow} \mathbf{N}^{I(x)}$ and the images of the generators of $\overline{M}_x$ in $\mathcal{O}_X$, which are well defined up to units, define the subschemes B_i locally at x.

2.1.3. *Pullback and restriction.* Given a morphism $f : Y \to X$ and a boundary $B = \{B_i\}_{i \in I}$ on X we define the pullback $f^*(B)$ as the tuple of all pullbacks $f^*(B_i) := B_i \times_X Y$ (which are locally principal in an obvious way). In the case when f is an immersion we will sometimes call to $f^*(B)$ the *restriction* of B on Y and denote it as $B|_Y$.

2.1.4. *B-schemes.* A *scheme with boundary* or simply *B-scheme* is a pair (X, B) consisting of a scheme X with a boundary B. We will say that (X, B) is *qe, of characteristic zero, generically reduced,* etc. if the scheme X is so. Note, however, that the notion of regular B-schemes will be defined below in a different way.

2.1.5. *Morphisms of B-schemes.* A *morphism $f : (X', B') \to (X, B)$ of B-schemes* is a morphism $f : X' \to X$ and a map $g : I \hookrightarrow \mathbf{N}^{I'}$ (which can also be interpreted as a map $g : I \times I' \to \mathbf{N}$ or a matrix $g \in \mathbf{N}^{I \times I'}$ with natural entries) such that $f^*(B_i) = \sum_{i' \in I'} g(i, i') B'_{i'}$ for each $i \in I$. We can shortly write this as $f^*(B) = gB'$.

If, moreover, g is induced by an ordered isomorphism $I \to I'$ (i.e. it is an identity matrix), and so $f^*(B) = B'$, then we say that the morphism is *exact* (borrowing this notion from log-geometry). Also, we say that f is *regular* if the morphism $X' \to X$ is.

Remark 2.1.3. (i) One easily sees that a morphism of B-schemes is nothing else but a morphism between the associated log-schemes.

(ii) It seems that the above interpretation justifies introducing of general B-morphisms, despite the fact that in this paper we will only be interested in exact regular morphisms and B-blow up sequences (see §2.2) which can (and will) be introduced in a simple ad hoc manner. Note that the B-blow ups are almost never exact.

2.1.6. *Snc and monomial boundaries.* A boundary B on X is called *snc* if each non-empty stratum B_J is regular and of pure codimension $|J|$ (including the stratum $B_\emptyset$). In particular, our definition is not local at $|B|$ because it implies that the entire X is regular. Similarly, we say that B is *strictly monomial* if X is regular and $|B|$ is snc.

Remark 2.1.4. The following conditions are equivalent:

(i) B is snc,

(ii) $[B]$ is an snc divisor and all components B_i are regular,

(iii) B is monomial, each B_i is a regular divisor (not necessarily connected) and no pair B_i, B_j with $i \neq j$ has a common irreducible component.

2.1.7. *Regular and semi-regular B-schemes.* A B-scheme (X, B) is *regular* if B is snc. Also, we will often use a slight weakening of the regularity condition. Namely, we say that a B-scheme (X, B) is *semi-regular* if locally at each point $x \in X$ the stratum $B_{\widetilde{I}(x)}$ is regular and of codimension $|\widetilde{I}(x)|$. Semi-regularity at x means that for a neighborhood of x we can split $B|_U$ as unordered disjoint union $B' \sqcup B''$ so that B' is snc and B'' consists of few copies of X.

2.1.8. *Regular and singular locus.* For a qe B-scheme (X, B) the set of points x at which (X, B) is semi-regular form an open subset which will be denoted $(X, B)_{\mathrm{sreg}}$. Its complement will be denoted $(X, B)_{\mathrm{ssing}}$ (that can be interpreted as strictly singular or severely singular locus) and will often be simply called the *bad locus* of (X, B). We will not make use of other regular/singular loci of (X, B).

2.2. **Blow up sequences and basic operations.**

In §2.2 we will study transforms of the boundaries under blow up sequences. One easily sees that the strict transform of a locally principal subscheme does not have to be locally principal, so the strict transform is useless in this context. Although for a boundary $B = \{B_i\}_{i \in I}$ on X and a blow up $f : \mathrm{Bl}_V(X) \to X$ the full transform $f^*(B) = \{f^*(B_i)\}_{i \in I}$ is defined, it is not the transform one usually uses. For example, there is a certain redundancy in the fact that many components of $f^*(B)$ can contain the exceptional divisor $E_f = f^*(V)$. Usually one tries at least to some extent to split off redundant copies of E_f and this leads to the definitions of principal and complete transforms which we give below. Also, in view of these definitions it will be natural to define exceptional divisor of a blow up sequence as a boundary (rather than a single divisor) which measures the difference between the transforms.

2.2.1. *Principal transform of closed subschemes.*

Remark 2.2.1. An important role in embedded resolution of singularities is played by a so called principle (weak or controllable) transform of ideals or marked ideals under blow ups. It is obtained from the full (or total) transform $f^*(Z)$ by removing an appropriate multiple (depending on the setting) of the exceptional divisor. Thus, the principal transform is a small step from the full transform towards the strict one, which still can be easily described by explicit formulas in terms of the corresponding ideals.

Let $f : X' \to X$ be the blow up along $V \hookrightarrow X$ and let $Z \hookrightarrow X$ be a closed subscheme. For our needs it will be convenient to adopt the following variant of principal transform of Z under f. Decompose V as $U \sqcup W$, where U is the union of all connected components of V that are closed subschemes in Z. Then the exceptional divisor $V' = f^*(V)$ possesses a component $U' = f^*(U)$ which is contained in $f^*(Z)$, and hence the closed subscheme $f^*(Z) - U'$ is defined. We call it the principal transform of Z and denote as $f^{\triangleright}(Z)$. A *principal transform* $g^{\triangleright}(Z)$ with respect to a blow up sequence g is defined iteratively.

Remark 2.2.2. In sharp contrast with strict and full transforms, the principal transform is not local on the base because it can happen that V is connected and is not a subscheme of Z (and so $f^{\triangleright}(Z) = f^*(Z)$), but $\emptyset \neq V|_U \hookrightarrow Z|_U$ for some open subscheme $U \hookrightarrow X$. The complete transform which will be defined later is also of non-local nature.

2.2.2. *Principal transform of the boundary.* Note that the principal transform of any locally principal closed subscheme is a locally principal closed subscheme. Therefore given a B-scheme (X, B) with $B = \{B_i\}_{i \in I}$ and a blow up sequence $f : X' \dashrightarrow X$ we can define the *principal transform* $f^{\triangleright}(B)$ as the tuple $\{f^{\triangleright}(B_i)\}_{i \in I}$. By the very definition, the principal transform is compatible with compositions of blow up sequences.

Remark 2.2.3. An equivalent definition of the principal transform is given in [CJS, 4.4], where the transform is called "principal strict transform". This terminology seems to be slightly unprecise because there might be smaller principal transforms containing the strict transform (e.g. when we modify the definition as $f^{\square}(Z) = f^*(Z) - \sum_i n_i f^*(V_i)$ where V_i's are the connected components of the center of f and n_i is the maximal number for which $n_i f^*(V_i)$ is a subscheme of $f^*(Z)$).

2.2.3. *Complete transform of the boundary.* Assume that (X, B) is as above and $X' = \mathrm{Bl}_V(X)$. Then we define the *complete transform* of the boundary $f^{\circ}(B) = f^{\triangleright}(B) \sqcup_{ord} \{E_f\}$, where $E_f = V \times_X X'$ is the *exceptional divisor* of the blow up along V. Note that E_f depends on the blow up and is not determined only by the morphism $X' \to X$, and we adjoin E_f as a new element even when $f^{\triangleright}(B)$ already contains some copies of E_f. The main reason for introducing the complete transform is that $|f^*(B)| \subseteq |f^{\circ}(B)| = |f^*(B)| \cup |E_f|$, because $[f^{\circ}(B)] = [f^{\triangleright}(B)] + E_f = (f^*(B) - nE_f) + E_f$ where n is a number for which $f^*(B) - nE_f$ is defined. We aware the reader that unlike the principal transform, the complete transform is not additive, i.e. $f^{\circ}(B \sqcup B') \neq f^{\circ}(B) \sqcup f^{\circ}(B')$ even as unordered sets. If $f : X' \dashrightarrow X$ is a general blow up sequence then we define the complete transform $f^{\circ}(B)$ iteratively. In particular, $f^{\circ}(B)$ is the ordered disjoint union of the *old boundary* $f^{\triangleright}(B)$ and a *new boundary* E_f which we also call the *boundary of the blow up sequence f*.

Example 2.2.4. (i) Assume that $V = B_i \in B$ is a Cartier divisor whose connected components are not contained in any B_j with $j \neq i$. Then $X' \widetilde{\rightarrow} X$ and $f^\circ(B)$ equals to B as unordered sets. However, the order is different because we move B_i to be the largest element in the boundary. Indeed, $f^\triangleright(B_i) = \emptyset$ and so we remove the i-th component, but $E_f = B_i$ and so we adjoin the same component with the largest index.

(ii) If $B_i = nV$ then the i-th boundary component disappears after n blow ups with center at V.

2.2.4. *Summary of transforms.* For the sake of referencing we collect basic properties of the transforms in the following Lemma. Since the strict transform $f^!(B)$ is not defined (at least as a boundary), we will consider $f^!(|B|)$ and $f^!([B])$ instead.

Lemma 2.2.5. *Let $f : X' \dashrightarrow X$ be a blow up sequence with new boundary E_f and let $B = \{B_1, \ldots, B_n\}$ be a boundary on X, then*

(i) $|E_f|$ is the reduced exceptional divisor, i.e. $|E_f|$ is the smallest closed subset of X' such that $X' \setminus |E_f| \widetilde{\rightarrow} X \setminus f(|E_f|)$,

(ii) the total and principal transforms are componentwise in the sense that $f^*(B) = \{f^*(B_1), \ldots, f^*(B_n)\}$ and $f^\triangleright(B) = \{f^\triangleright(B_1), \ldots, f^\triangleright(B_n)\}$, and the complete transform is obtained from the principal transform by adjoining E_f, i.e. $f^\circ(B) = f^\triangleright(E) \sqcup_{ord} E_f$,

(iii) we have componentwise inclusions of strict, principal and full transforms $f^!(B_i) \hookrightarrow f^\triangleright(B_i) \hookrightarrow f^*(B_i)$, where the last two components are always locally principal and in addition $f^*(B_i) = f^\triangleright(B_i) + D_i$, where D_i is an exceptional divisor (i.e. $|D_i| \subset |E_f|$),

(iv) on the level of supports the transforms are related as follows: $|f^!(|B|)| \subseteq |f^\triangleright(B)| \subseteq |f^*(B)| \subseteq |f^\circ(B)| = |f^!(|B|)| \cup |E_f|$,

(v) on the level of divisorial supports the transforms are related as follows: $f^![B] \hookrightarrow [f^\triangleright(B)] \hookrightarrow [f^*(B)]$ and $[f^\triangleright(B)] + [E_f] = [f^\circ(B)]$.

(vi) Our principal transform of closed subschemes agrees with the principal transform of marked ideals of order one. More concretely, assume that $\underline{\mathcal{I}} = (N, N, \emptyset, \mathcal{I}, 1)$ is a marked ideal with an admissible blow up sequence $f : N' \dashrightarrow N$ and let $\underline{\mathcal{I}}' = (N', N', E', \mathcal{I}', 1)$ be the transform of $\underline{\mathcal{I}}$ (see [BM3, §2]). Then $Z' = f^\triangleright(Z)$ where $Z' \hookrightarrow N'$ and $Z \hookrightarrow N$ are the closed subschemes defined by $\mathcal{I}'$ and $\mathcal{I}$. In particular, if f is a resolution of $\underline{\mathcal{I}}$ then $f^\triangleright(Z) = \emptyset$.

Proof. The assertions (i)–(vi) are easily verified by induction on the length (and many statements just repeat the definitions). We will only prove (vi) to illustrate this. Let V be the first center of f. Then V is in the cosupport of $\underline{\mathcal{I}}$ which is precisely $|Z|$, and hence $Z_1 = f^\triangleright(Z) = f^*(Z) - f^*(V)$. On the other hand, $\mathcal{I}_1 = \mathcal{I}_E^{-d}(\mathcal{I}\mathcal{O}_{N'})$ where $E = f^*(V)$ is the exceptional divisor and $d = 1$ is the order of $\underline{\mathcal{I}}$. This implies the claim for a length one sequence and we deduce that $\text{cosupp}(\underline{\mathcal{I}}_1) = |Z_1|$ contains the second center of f. So, we can repeat the argument for the second blow up, etc., thus mastering induction on the length. $\square$

Remark 2.2.6. Typically, one operates with principal and complete transforms when building a desingularization functor, and a desingularization is often achieved by getting empty $f^\triangleright(B)$ and snc $f^\circ(B)$. It then follows from Lemma 2.2.5(iv) that $f^!(|B|)$ is empty and $f^*(B)$ is strictly monomial.

2.2.5. *B-blow up sequences.* In the context of desingularization, B-schemes are introduced in order to control the boundaries. In particular, it is important to control the boundary of the blow up sequences (i.e. the exceptional divisors). So, we define the *B-blow up* $F : (X', B') \to (X, B)$ as a blow up $f : X' \to X$ such that $B' = f^\circ(B)$. By the *center* of F we mean the center of f and say that F is *trivial* if f is (i.e. the center is empty).

Remark 2.2.7. (i) Unlike blow ups of schemes, usually a B-blow up f is not an isomorphism of B-schemes even when its center V is a Cartier divisor. Actually, one easily sees that f is an isomorphism if and only if either f is trivial or we are in the situation of Example 2.2.4(i) and in addition $V = B_i$ is the largest element of the boundary (so we erase the largest component of the boundary and then adjoin it again).

(ii) Although we will not use this in the paper, we note that a B-blow up possesses a natural structure of a morphism of B-schemes. Indeed, for each $B_i \in B$ we have that $f^*(B_i) = f^\triangleright(B_i) + nE_f$, where $n \in \{0, 1\}$ and both $f^\triangleright(B_i)$ and E_f are components of $B' = f^\circ(B)$.

Naturally, a *B-blow up sequence* $F : (X', B') \dashrightarrow (X, B)$ is a sequence of B-blow ups. To give such a sequence with target (X, B) is equivalent to give a blow up sequence $f : X' \dashrightarrow X$ because the boundaries are uniquely determined as complete transforms. Therefore, given a B-scheme (X, B) we will pass freely between blow up sequences of X and B-blow up sequences of (X, B).

2.2.6. *Restriction of transforms onto closed subschemes.* Assume that $f : X' \to X$ is a blow up along V and $Z \hookrightarrow X$ is a closed subscheme. An easy explicit computation on the charts of $\mathrm{Bl}_V(X)$ shows that the strict transform $Z' = f^!(Z)$ is the blow up of Z along $\widetilde{V} = V|_Z$, see [Con, §1]. This also implies that the restriction of the exceptional divisor $V \times_X X'$ onto Z' is the exceptional divisor $\widetilde{V} \times_Z Z'$ of the blow up $\widetilde{f} : Z' \to Z$. Assume now that B is a boundary on X and $\widetilde{B} = B|_Z$ is its restriction on Z. By transitivity of fibred products we obviously have that $(B \times_X X')|_{Z'} = \widetilde{B} \times_Z Z'$ but the other transforms of the boundary do not have to be compatible with the restriction. Furthermore, it is clear that the equalities $f^\triangleright(B)|_{Z'} = \widetilde{f}^\triangleright(\widetilde{B})$ and $f^\circ(B)|_{Z'} = \widetilde{f}^\circ(\widetilde{B})$ fail if and only if there exists $B_i \in B$ and a connected component $V' \hookrightarrow V$ such that V' is not a subscheme of B_i but $V'|_Z$ is a subscheme of $B_i|_Z$, and so we compute the transforms under f and $\widetilde{f}$ using different cases. We observe, however, that this bad situation cannot occur whenever $V \hookrightarrow Z$, so we at least have the following Lemma.

Lemma 2.2.8. *Let X be a scheme with a closed subscheme Z and assume that $f : (X', B') \dashrightarrow (X, B)$ is B-blow up sequence whose centers are closed subschemes of the strict transforms of Z. Denote by $\widetilde{f} : Z' \dashrightarrow Z$ the induced blow up sequence of strict transforms. Then $f^\triangleright(B)|_{Z'} = \widetilde{f}^\triangleright(\widetilde{B})$ and $f^\circ(B)|_{Z'} = \widetilde{f}^\circ(\widetilde{B})$. In particular, $(Z', B'|_{Z'}) \dashrightarrow (Z, B|_Z)$ is a B-blow up sequence which will be denoted $f|_Z$.*

2.3. **Permissible B-blow up sequences.**

2.3.1. *Transversality to the boundary.* Given a B-scheme (X, B) and a closed subscheme $Z \hookrightarrow X$ we say that Z is *transversal to the boundary* if $(Z, B|_Z)$ is a regular B-scheme. A more traditional way to formulate this condition is to say that each scheme $Z \times_X B_{i_1} \times_X \ldots B_{i_n}$ is either empty or regular of codimension n in Z. In

particular, taking $n = 0$ we see that Z itself is regular. In the important particular case when (X, B) is regular this just means that Z is regular and transversal to the snc divisor $|B|$ in the usual sense.

2.3.2. *Simple normal crossings with the boundary.* More generally, we say that Z has *simple normal crossings with the boundary* if $(Z, B|_Z)$ is a semi-regular B-scheme. As earlier, in the case when (X, B) is regular this reduces to the usual notion of being regular and having simple normal crossings with an snc divisor.

2.3.3. *Permissibility.* A B-blow up $(X', B') \rightarrow (X, B)$ is called *permissible* if its center $V \hookrightarrow X$ has simple normal crossings with the boundary B. More generally, a B-blow up sequence is *permissible* if all its B-blow ups are permissible.

Remark 2.3.1. (i) Even when B is empty the permissibility condition is stronger than just blowing up along regular centers. Namely, the first center just has to be regular, but after the first blow up a non-empty boundary appears and this imposes an additional restriction on the choice of further centers.

(ii) For the sake of giving an inductive proof, even when interested in embedded desingularization without boundaries one often has to treat the exceptional divisors with a certain respect, and practically this amounts to considering only permissible blow up sequences. In particular, the known embedded desingularization algorithms for varieties construct permissible blow up sequences. A partial reason for this is seen in the following Lemma.

Lemma 2.3.2. *If (X, B) is a regular B-scheme and $(X', B') \dashrightarrow (X, B)$ is a permissible B-blow up sequence then (X', B') is regular and $f^{\triangleright}(B) = f^!(B)$.*

Proof. We can assume that f is a single blow up along V. Let B_i be a component of B and V_0 be a connected component of V. It is easy to see that both in the case when V_0 is transversal to B_i and in the case when V_0 is contained in B_i one has that $f^{\triangleright}(B_i) = f^!(B_i)$ in a neighborhood of $f^*(V_0)$. So, $f^{\circ}(B) = f^!(B) \sqcup_{ord} E_f$ and it is well known that the latter is an snc divisor whenever V has simple normal crossings with B. $\qquad\qquad\square$

2.3.4. *B-permissibility.* Often it will be convenient to express the permissibility in terms of usual blow up sequences and the initial boundary B. So, given a B-scheme (X, B) we say that a blow up sequence $X' \dashrightarrow X$ is *B-permissible* if the induced B-blow up sequence $(X', B') \dashrightarrow (X, B)$ is permissible.

2.3.5. *Pushforward and restriction.*

Lemma 2.3.3. *Assume that X is a scheme with a boundary B and closed sub-scheme $i : \widetilde{X} \hookrightarrow X$, and set $\widetilde{B} = B|_{\widetilde{X}}$.*

(i) If $\widetilde{f} : \widetilde{X}' \dashrightarrow \widetilde{X}$ is a $\widetilde{B}$-permissible blow up sequence then the pushforward $f = i_*(\widetilde{f})$ is a B-permissible blow up sequence.

(ii) If $f : X' \dashrightarrow X$ is a B-permissible blow up sequence whose centers are contained in the strict transforms of $\widetilde{X}$ then the induced blow up sequence of strict transforms $\widetilde{f} : f^!(\widetilde{X}) \dashrightarrow \widetilde{X}$ (see [Tem2, §2.2.7]) is $\widetilde{B}$-permissible.

Proof. Note that both in (i) and (ii) the centers of f lie on the strict transforms of $\widetilde{X}$, and hence the transforms of the boundaries in the blow up sequences f and $\widetilde{f}$ are compatible with the restriction by Lemma 2.2.8. Now, an obvious induction

on the length of f reduces the proof of both (i) and (ii) to the claim that if both $\widetilde{f}$ and f is a single blow up along $V \hookrightarrow \widetilde{X}$ then either f is B-permissible and $\widetilde{f}$ is $\widetilde{B}$-permissible or both blow ups are not permissible. This follows from the simple fact that for any connected component V_0 of V we have that either V_0 is contained in both B_i and $\widetilde{B}_i = B_i|_{\widetilde{X}}$, or V_0 is transversal to both B_i and $\widetilde{B}_i$ (we use that $V \hookrightarrow \widetilde{X}$ and so $B_i|_V = \widetilde{B}_i|_V$). $\square$

2.4. Desingularization of generically reduced B-schemes.

2.4.1. *Desingularization of a B-scheme.* By *desingularization* of a generically reduced B-scheme (X, B) we mean a B-blow up sequence $f : (X', B') \dashrightarrow (X, B)$ such that the B-scheme (X', B') is semi-regular and the centers of f are disjoint from the preimages of $(X, B)_{\text{sreg}}$. If, in addition, the centers of f are regular (resp. B-permissible) then we say that the desingularization is *strong* (resp. *B-strong*).

Remark 2.4.1. (i) Using a desingularization $f : (X', B') \dashrightarrow (X, B)$ as above one can obviously produce a B-blow up sequence that modifies the non-regular locus of (X, B) and produces a regular B-scheme (X'', B'') – one just has to kill the appropriate components of X' by blowing them up.

(ii) Also, it is easy to produce from f a B-blow up sequence $g : (X'', B'') \to (X, B)$ such that (X'', B'') is semi-regular, $g^{\triangleright}(B) = \emptyset$ and the centers of g sit over $|B| \cup X_{\text{sing}}$. This is done by blowing up the strata of B' starting with the smallest ones. Namely, we blow up B'_I at the first stage. This resolves the strict transform of $\cup_{i \in I} B'_{I \setminus \{i\}}$, so we can blow it up at the second stage and proceed similarly until the strict transform of $\cup_{i \in I} B'_i$ is blown up at the last stage. The new sequence g, which is obtained by extending f in this way, is as required.

(iii) The definition of desingularization makes sense for any B-scheme, but it is not the "interesting" definition when X is not generically reduced (see [Tem2, §1.2.2]). Such stupid desingularization will be used only in §3.5.2, where we will easily construct it from desingularization of generically reduced B-schemes.

2.4.2. *Functoriality.* Let $\mathfrak{C}$ be a category whose objects are certain generically reduced B-schemes and whose morphisms are certain exact regular morphisms. Then a *functorial desingularization* on $\mathfrak{C}$ is a rule $\mathcal{F}$ which associates a desingularization $\mathcal{F}(X, B) : (X', B') \dashrightarrow (X, B)$ to each B-scheme (X, B) from $\mathfrak{C}$ in a way *compatible* with the morphisms of $\mathfrak{C}$. The latter means that for each $h : (\overline{X}, \overline{B}) \to (X, B)$ in $\mathfrak{C}$ the B-blow up sequence $\mathcal{F}(\overline{X}, \overline{B})$ is obtained from $h^*(\mathcal{F}(X, B))$ by omitting all empty B-blow ups.

2.4.3. *The case of varieties.* Let $\mathcal{F}_{\text{Var}}$ be the non-embedded desingularization functor from [BMT, Th. 6.1]. Recall that it associates to a variety X of characteristic zero a strong desingularization $\mathcal{F}(X) : X' \dashrightarrow X$ and is compatible with all regular morphisms. Moreover, the addendum to [BMT, Th. 6.1] asserts that the associated B-blow up sequence $(X', B') \dashrightarrow (X, \emptyset)$ is a B-strong desingularization. In other words this can be formulated as follows.

Theorem 2.4.2. *There exists functorial B-strong desingularization $\mathcal{F}_{\text{Var}}$ on the category $\text{BVar}_{B=\emptyset}$ whose objects are finite disjoint unions of generically reduced B-varieties of characteristic zero with empty boundary and whose morphisms are all regular morphisms.*

2.5. **Formal and analytic analogs.** All definitions concerning boundaries, B-schemes, B-blow ups, desingularization of B-schemes and functoriality apply almost verbatim to the contexts of qe formal schemes and complex or non-archimedean analytic spaces. Note that semi-regularity is preserved by the following functors: (1) formal completion of qe S-schemes along a fixed ideal on a base scheme S, (2) analytification of k-varieties where k is a complete field (either archimedean or non-archimedean). In particular, it follows that the completion and analytification functors take desingularizations of B-schemes to desingularizations of formal B-schemes or analytic B-spaces.

3. DESINGULARIZATION OF QE B-SCHEMES OF CHARACTERISTIC ZERO

3.1. B-**Varieties.** Let BVar be the category of finite disjoint unions of generically reduced B-varieties of characteristic zero with all regular exact morphisms between them.

Theorem 3.1.1. *There exists a strong desingularization functor* $\mathcal{F}_{\mathrm{BVar}}$ *on the category* BVar.

Proof. We will construct a strong desingularization $\mathcal{F}_{\mathrm{BVar}}(X, B)$ of an object (X, B) from BVar, and the functoriality will be clear from the construction. Also, we use induction on $d = \dim(X)$, and the case of $d = 0$ is trivial (just take the empty blow up).

Step 1. *We can assume that X is regular and each B_i is a Cartier divisor.* We simply apply to X the strong non-embedded desingularization functor from [BMT, Theorem 6.1]. This gives a strong desingularization $X' \dashrightarrow X$, which we extend to a B-blow up sequence $f : (X', B') \dashrightarrow (X, B)$. By regularity of X', each component of $B' = \{B'_1, \ldots, B'_n\}$ decomposes as $B'_i = \overline{B}'_i \sqcup \widetilde{B}'_i$ where $\overline{B}'_i$ is a Cartier divisor and $\widetilde{B}'_i$ is a component of X' (in the sense of §1.2.4) . Clearly, any strong desingularization $X'' \dashrightarrow X'$ of $(X', \overline{B}')$ is also a strong desingularization of (X', B') and hence the composition $X'' \dashrightarrow X$ is a strong desingularization of (X, B). So, our problem reduces to desingularizing $(X', \overline{B}')$ with regular X' and divisorial boundary.

In the sequel we assume that (X, B) is as in Step 1 and we will construct a strong desingularization $f : (X', B') \dashrightarrow (X, B)$. This will be done by composing few B-blow up sequences which will gradually improve (X', B'). To simplify the notation some intermediate B-blow up sequences will be also denoted by f, and we will update f from time to time by composing it with a new sequence $(X'', B'') \dashrightarrow (X', B')$. All centers of our blow ups will be regular and Z-supported for $Z = (X, B)_{\mathrm{ssing}}$ (i.e. they will be contained in the preimage of Z). In particular, we will always have that the conditions of Step 1 are satisfied and $Z' = (X', B')_{\mathrm{ssing}}$ is Z-supported, and so we can blow up any regular subvariety of Z'. Also, we will use the following notation: $B = \{B_1, \ldots, B_n\}$, $B'_i := f^{\triangleright}(B_i)$, $B^!_i = f^!(B_i)$ and $B^! = \{B^!_1, \ldots, B^!_n\}$. Note that $B' = \{B'_1, \ldots, B'_n\} \sqcup_{ord} E'$, where E' is the boundary of f.

Remark 3.1.2. Now, a straightforward attempt to desingularize (X, B) would be to iteratively apply embedded desingularization functor to the components of B. More concretely, if $f_1 : X_1 \dashrightarrow X$ resolves the marked ideal $(X, X, \emptyset, B_1, 1)$ in the sense of [BM3, §5] or [BMT, §5] then $f_1^{\triangleright}(B_1) = \emptyset$ by Lemma 2.2.5(vi) and

hence $E_1 := f_1^\circ(B_1)$ is an snc divisor. After that one could take $f_2 : X_2 \dashrightarrow X_1$ that resolves $(X_1, X_1, E_1, f_1^\triangleright(B_2), 1)$, etc., obtaining in the end a blow up sequence $f : X' \dashrightarrow X$ with snc $f^\circ(B)$ and empty $f^\triangleright(B)$. This method is not fine enough because it modifies the entire $|B|$ rather than the bad locus Z. A possible refinement of this method is given in Remark A.1.1(iii), but we prefer here yet another method.

Step 2. *We can achieve that $|B_1^!|$ is disjoint from Z'.*

Substep (a). *It is enough to construct Z'-supported B-blow up sequence $f :$ $(X', B') \dashrightarrow (X, B)$ such that $B_1^!$ is disjoint from Z'.* Indeed, if this is the case then we can similarly construct a Z'-supported blow up sequence which separates the strict transform of B_2 from the bad locus. Since Z' is disjoint from $B_1^!$, this will not affect the situation at $B_1^!$, and so the new sequence separates the strict transforms of both B_1 and B_2 from the bad locus. So, repeating this procedure few times we would accomplish Step 2.

In the sequel we will only construct f as in Substep (a). Also, by $\widetilde{X}$ we denote the reduction of B_1.

Substep (b). *General plan.* We will construct f by composing few blow up sequences and for simplicity we will denote all intermediate sequences as $f : X' \dashrightarrow$ X. While constructing f we will only blow up the centers lying on the strict transforms of $\widetilde{X}$, and so f is the pushforward of the induced blow up sequence $\widetilde{f} : \widetilde{X}' \dashrightarrow \widetilde{X}$, where $\widetilde{X}' = f^!(\widetilde{X})$. Also, we will use the notation $\widetilde{B} = (B \setminus \{B_1\})|_{\widetilde{X}}$ and $\widetilde{B}' = \widetilde{f}^\circ(\widetilde{B})$.

Substep (c). *We can achieve that the B-scheme $(\widetilde{X}', \widetilde{B}')$ is semi-regular.* The dimension of $\widetilde{X}$ is strictly smaller than that of X and hence the desingularization $\widetilde{f} = \mathcal{F}_{\mathrm{BVar}}(\widetilde{X}, \widetilde{B}) : (\widetilde{X}', \widetilde{B}') \to (\widetilde{X}, \widetilde{B})$ exists by the induction assumption. So, we accomplish this substep by letting $f : (X', B') \dashrightarrow (X, B)$ be the pushforward of $\widetilde{f}$ with respect to the closed immersion $\widetilde{X} \hookrightarrow X$. Indeed, obviously $\widetilde{X}' = f^!(\widetilde{X})$ and $B'|_{\widetilde{X}'} = \widetilde{B}'$ by Lemma 2.2.8

Substep (d). *We can achieve that $\widetilde{X}' = B_1^!$ and $(\widetilde{X}', \widetilde{B}')$ is regular.* Clearly, $\widetilde{X}'$ is the reduction of $B_1^!$ and any non-reduced component of $B_1^!$ is contained in Z'. So, we can simply blow up the components of $\widetilde{X}'$ that underly a non-reduced component of $B_1^!$, thereby achieving that $\widetilde{X}' = B_1^!$. Similarly, to make $(\widetilde{X}', \widetilde{B}')$ regular we should blow up the components of $\widetilde{X}'$ that are contained in $\widetilde{B}'$. Any such component is contained in at least two boundary components of B' (these are B_1' and some component of $f^\circ(B \setminus \{B_1\})$). So, any such component is in Z' and we can safely blow it up. (Actually, both blow ups of this Substep can be done simultaneously.)

Substep (e). *In addition to the above we can achieve that $\widetilde{X}'$ is a component of $f^\triangleright(B_1)$.* Since X' is regular, $\widetilde{X}'$ is a divisor and hence we obtain a splitting $f^\triangleright(B_1) = \widetilde{X}' + Y'$ where Y' is a divisor with $|Y'| \subset |E_f|$. To accomplish the Substep we should make Y' disjoint from $\widetilde{X}'$ and this is achieved if and only if the divisor $\widetilde{Y}' := Y'|_{\widetilde{X}'}$ on $\widetilde{X}'$ is empty. Note that $|\widetilde{Y}'| \subset |\widetilde{X}'| \cap |E_f| \subset |\widetilde{B}'|$. In particular, $\widetilde{Y}'$ is snc and there exist minimal numbers $m_1, \ldots, m_k$ such that $\widetilde{Y}' \hookrightarrow \sum_{i=1}^k m_i D_i$, where $\widetilde{B}' = \{D_1, \ldots, D_k\}$. We will kill $\widetilde{Y}'$ by few simple blow ups as follows. Take the first non-zero m_j and consider the blow ups $g : X'' = \mathrm{Bl}_{D_j}(X') \to X'$ and $X''' = \mathrm{Bl}_{E''}(X'') \to X'$, where $E'' = g^*(D_j)$ is the exceptional divisor of g. Both blow ups have regular centers and are D_j-supported and hence Z-supported. Also,

they do not modify $\widetilde{X}'$ (i.e. $\widetilde{X}''' \widetilde{\to} \widetilde{X}'$). Finally, a straightforward computation with charts shows that if we replace $X' \dashrightarrow X$ with the composition $X''' \dashrightarrow X$ then we achieve that the new $\widetilde{Y}'$ is contained in $(\sum_{i=1}^k m_i D_i) - D_j$. Thus, it remains to repeat the same operation few more times, until $\widetilde{Y}'$ vanishes. For the sake of completeness, the chart computation will be done in Lemma 3.1.3 after the proof of the Theorem.

Substep (f). *The conditions from substeps (d) and (e) imply that $\widetilde{X}'$ is disjoint from the bad locus of (X', B'), and this accomplishes Step 2.* Obviously, $f^\circ(B) = \{f^\triangleright(B_1)\} \sqcup f^\circ(B \setminus \{B_1\})$, hence in a neighborhood of $\widetilde{X}'$ this boundary coincides with $\{\widetilde{X}'\} \sqcup f^\circ(B \setminus \{B_1\})$. Since X' is a regular scheme with a regular divisor $\widetilde{X}' = B_1^!$ and the restriction of $f^\circ(B \setminus \{B_1\})$ onto $\widetilde{X}'$ is an snc divisor (it is $\widetilde{B}'$), we have that $f^\circ(B)$ is an snc divisor in a neighborhood of $\widetilde{X}'$.

Step 3. *Resolution of B.* Obviously, $D' = [B'] - [B^!]$ is a Z-supported divisor. Let $g : X'' \dashrightarrow X'$ be the resolution of the marked ideal $(X', X', B^!, D', 1)$ as defined in [BMT, §5] (we use that $B^!$ is snc after Step 2). Thus, g depends functorially on (X, B) and is D'-supported and hence Z-supported. Also, $g^\triangleright(D') = \emptyset$ by Lemma 2.2.5(vi), and $g^\triangleright(B^!) \sqcup_{ord} E_g$ is snc, where E_g is the boundary of g. Since $B_i' = B_i^! + \overline{B}_i$ where $\overline{B}_i \hookrightarrow D'$ and $B_i^!$ is disjoint from $\overline{B}_i$, we obtain that $g^\triangleright(B_i') = g^\triangleright(B_i^!)$. Therefore, $g^\circ(B') = g^\triangleright(B^!) \sqcup_{ord} E_g$ is snc, and so the composition $X'' \dashrightarrow X$ is a strong desingularization of (X, B). $\square$

Now let us prove the simple Lemma used earlier.

Lemma 3.1.3. *Assume that X is a regular scheme with a regular divisor Y and Z is a divisor in X such that $Z|_X = \sum_{i=1}^n m_i D_i$, where $\{D_1, \ldots, D_n\}$ is an snc boundary on Y. Let $f : X'' \to X' \to X$ be obtained by first blowing up D_1 and then blowing up $D_1' = D_1 \times_X X'$. Then $Y'' = f^!(Y)$ is isomorphic to Y and the restriction of $f^\triangleright(Y + Z)$ onto Y'' is isomorphic to $m_1' D_1 + \sum_{i=2}^n m_i D_i$, where $m_1' = \max(0, m_1 - 1)$.*

Proof. It is enough to check the claim locally at any point $p \in X$. Working étale-locally we can assume that $X = \mathbf{A}^{n+1} = \mathrm{Spec}(k[x, y])$, $Y = V(x)$ and $D_i = V(x, y_i)$, where $y = (y_1, \ldots, y_n)$ and p is some point in X. Then $Z = V(\phi)$, where $\phi = y_1^{m_1} \ldots y_n^{m_n} + xP(x, y)$ and after the blow up along D_1, which we denote as $g : X' \to X$, we obtain that ϕ becomes $\phi' = y_1^{m_1} \ldots y_n^{m_n} + y_1 x' P(y_1 x', y)$ in the new coordinate system $(x' = \frac{x}{y_1}, y_1, \ldots, y_n)$. In particular, $g^*(Y + Z) = V(y_1 \phi')$ and $T = g^\triangleright(Y + Z) = V(\phi')$. The second blow up is an identity $h : X'' \widetilde{\to} X'$ and its transform is computed as follows. If $m_1 = 0$ then $D_1' = V(y_1) \not\subseteq T$ and so $h^\triangleright(T) = h^*(T) = T$; if $m_1 = 0$ then $D_1' \subseteq T$ and so $h^\triangleright(T) = T - D_1'$. In the first case $f^\triangleright(Y + Z) = T$ is defined by ϕ', and in the second case, $f^\triangleright(Y + Z) = T - D_1$ is defined by the element $\phi'/y_1 = y_1^{m_1-1} y_2^{m_2} \ldots y_n^{m_n} + x'P(y_1 x', y)$. The Lemma follows. $\square$

3.2. **Formal B-varieties with small singular locus.**

3.2.1. *Locally principal formal B-schemes.* By a *(locally) principal formal B-scheme* we mean a triple $(\mathfrak{X}, \mathfrak{B}, \mathfrak{I})$ where $(\mathfrak{X}, \mathfrak{B})$ is a formal B-scheme and $\mathfrak{I}$ is an invertible ideal of definition of $\mathfrak{X}$. Sometimes we will replace $\mathfrak{I}$ with $\mathfrak{Z} = \mathrm{Spf}(\mathcal{O}_\mathfrak{X}/\mathfrak{I})$ in this notation. By a morphism $(\mathfrak{X}', \mathfrak{B}', \mathfrak{Z}') \to (\mathfrak{X}, \mathfrak{B}, \mathfrak{Z})$ of such creatures we always mean a morphism of formal B-schemes $f : (\mathfrak{X}', \mathfrak{B}') \to (\mathfrak{X}, \mathfrak{B})$ such that $\mathfrak{Z}' = f^*(\mathfrak{Z})$.

Actually, we will only be interested in the cases when f is either an exact regular morphism or a formal B-blow up sequence.

Remark 3.2.1. (i) Both $\mathfrak{Z}$ and the components of $\mathfrak{B}$ are locally principal closed formal subschemes. However, they transform differently under formal B-blow ups. This is the reason to distinguish $\mathfrak{Z}$ rather than to include it as a special (e.g. minimal) component of the boundary.

(ii) The role of $\mathfrak{Z}$ will be to contain (and control) the bad locus of the formal B-scheme $(\mathfrak{X}, \mathfrak{B})$.

3.2.2. *A category* $\widehat{\mathrm{BVar}}_{\mathrm{small}}$. We now introduce a category $\widehat{\mathrm{BVar}}_{\mathrm{small}}$ whose objects are finite disjoint unions of certain locally principal formal varieties with small bad locus. More concretely, $(\mathfrak{X}, \mathfrak{B}, \mathfrak{Z})$ is in $\widehat{\mathrm{BVar}}_{\mathrm{small}}$ if $(\mathfrak{X}, \mathfrak{B})$ is a finite disjoint union of formal B-varieties of characteristic zero, $\mathfrak{B}$ is $\mathfrak{Z}$-supported (i.e. all components of $\mathfrak{B}$ are supported on the closed fiber $\mathfrak{X}_s$) and $\mathfrak{X}$ is rig-regular. The morphisms in $\widehat{\mathrm{BVar}}_{\mathrm{small}}$ are all exact regular morphisms between its objects.

Remark 3.2.2. (i) When $\mathfrak{B}$ is empty we obtain the category $\widehat{\mathrm{Var}}_{p=0}$ from [Tem2, §3.1].

(ii) Assume that $(\mathfrak{X}, \mathfrak{B}, \mathfrak{Z})$ is an object of $\widehat{\mathrm{BVar}}_{\mathrm{small}}$ and $(X, \mathcal{I})$ is an algebraization of $(\mathfrak{X}, \mathfrak{Z})$ in the sense of [Tem2, §3.1]. Since $\mathfrak{B}$ is $\mathfrak{Z}$-supported it algebraizes uniquely to an $\mathcal{I}$-supported boundary B on X and $B_i = \mathfrak{B}_i$ as schemes. In particular, this algebraization uniquely extends to an algebraization $(X, B, \mathcal{I})$ of the original triple. Moreover, all components of B are closed subschemes in the n-th fibers $X_n = \mathrm{Spec}(\mathcal{O}_X/\mathcal{I}^n)$ for large enough n. It follows that all results from [Tem2, §3.2] obviously generalize to the objects of $\widehat{\mathrm{BVar}}_{\mathrm{small}}$. For example, an analog of [Tem2, Prop. 3.2.1] holds true and the proof is almost identical with the only minor modification that one should always consider thick enough n-th fibers so that they contain B.

3.2.3. *Desingularization on* $\widehat{\mathrm{BVar}}_{\mathrm{small}}$. Now we are ready to generalize [Tem2, Th. 3.1.5] to formal B-schemes.

Theorem 3.2.3. *let $\mathcal{F}_{\mathrm{BVar}}$ be a desingularization functor on BVar. Then there exists unique up to unique isomorphism desingularization functor $\widehat{\mathcal{F}}_{\mathrm{BVar}}$ on $\widehat{\mathrm{BVar}}_{\mathrm{small}}$ such that $\widehat{\mathcal{F}}_{\mathrm{BVar}}$ is compatible with $\mathcal{F}_{\mathrm{BVar}}$ under formal completions. Moreover, $\widehat{\mathcal{F}}_{\mathrm{BVar}}$ is strong (resp. B-strong) if and only if $\mathcal{F}_{\mathrm{BVar}}$ is strong (resp. B-strong).*

Proof. The argument repeats the proof of [Tem2, Th. 3.1.5] (as given in [Tem2, §§3.2-3.3]) with the only minor modification that one should always consider thick enough n-th fibers so that they contain the boundary. $\square$

3.3. **B-schemes with small singular locus.** Our next aim is to generalize [Tem2, Th. 3.3.1] to B-schemes with small singular locus. Consider the category $\mathrm{BSch}_{\mathrm{small}}$ as follows. Objects of $\mathrm{BSch}_{\mathrm{small}}$ are triples (X, B, Z), where (X, B) is a generically reduced noetherian qe B-scheme of characteristic zero and $Z \hookrightarrow X$ is a Cartier divisor which is a disjoint union of varieties and contains $(X, B)_{\mathrm{ssing}}$. Morphisms $(X', B', Z') \to (X, B, Z)$ in $\mathrm{BSch}_{\mathrm{small}}$ are exact regular morphisms, i.e. regular morphisms $f : X' \to X$ such that $B' = f^*(B)$ and $Z' = f^*(Z)$.

Theorem 3.3.1. *Assume that there exists a strong desingularization functor $\widehat{\mathcal{F}}_{\mathrm{BVar}}$ on $\widehat{\mathrm{BVar}}_{\mathrm{small}}$. Then there exists a strong desingularization functor $\mathcal{F}_{\mathrm{small}}$ on $\mathrm{BSch}_{\mathrm{small}}$*

which assigns to a triple (X, B, Z) a desingularization of (X, B) and is compatible with all morphisms from $\mathrm{BSch}_{\mathrm{small}}$.

Proof. Let $T = (X, B)_{\mathrm{ssing}}$ denote the bad locus. Also, when possible we decompose the boundary B as a sum of inner and outer boundaries B_{in} and B_{out}, where $B_i = B_{i,\mathrm{in}} + B_{i,\mathrm{out}}$ for each $B_i \in B$, and the irreducible components of $|B_{i,\mathrm{in}}|$ are exactly the Z-supported irreducible components of B_i. When exists, this decomposition is unique, and it always exists when X is regular (or, more generally, locally factorial).

Case 1. *Empty outer boundary.* Let $\mathfrak{B}_0$ denote the full subcategory of $\mathrm{BSch}_{\mathrm{small}}$ whose objects have empty outer boundary (i.e. $|B| \subset Z$). We claim that $\widehat{\mathcal{F}}_{\mathrm{BVar}}$ induces a desingularization functor on $\mathfrak{B}_0$. Indeed, the formal completion of (X, B, Z) along Z is an object of $\widehat{\mathrm{BVar}}_{\mathrm{small}}$, and hence it is resolved by the functorial $\widehat{Z}$-supported B-blow up sequence $\widehat{\mathcal{F}}_{\mathrm{BVar}}(\widehat{X}, \widehat{B}, \widehat{Z})$. Similarly to the proof of [Tem2, 3.3.1] this sequence algebraizes to a functorial Z-supported B-blow up sequence $\mathcal{F}_0(X, B, Z) : X' \dashrightarrow X$ which provides a strong desingularization of (X, B). In particular, compatibility with all regular morphisms follows from [Tem2, 2.4.5].

Case 2. *Regular outer boundary.* Let $\mathfrak{B}_1$ denote the full subcategory of $\mathrm{BSch}_{\mathrm{small}}$ formed by the triples (X, B, Z) for which the decomposition $B = B_{\mathrm{in}} + B_{\mathrm{out}}$ is defined and $|B_{\mathrm{out}}|$ is disjoint from T. We claim that the functor $\mathcal{F}_0$ from Case 1 trivially extends to a desingularization functor $\mathcal{F}_1$ on $\mathfrak{B}_1$. Indeed, if (X, B, Z) is an object of $\mathfrak{B}_1$ then $(X, B_{\mathrm{in}})_{\mathrm{ssing}} \subset T$ is disjoint from $|B_{\mathrm{out}}|$ and it follows that $\mathcal{F}_0(X, B_{\mathrm{in}}, Z)$ resolves (X, B) (it obviously resolves (X, B) over $X \setminus |B_{\mathrm{out}}|$ and it does not change anything near $|B_{\mathrm{out}}|$). Thus, we simply set $\mathcal{F}_1(B, X, Z) = \mathcal{F}_0(X, B_{\mathrm{in}}, Z)$.

Case 3. *The general case.* Now, we are going to construct desingularization $\mathcal{F}_{\mathrm{small}}(X, B, Z)$ of a general object of $\mathrm{BSch}_{\mathrm{small}}$. We will use induction on the dimension of Z, so assume that $\dim(Z) = d$ and the functor is already constructed for smaller values of d. Our construction is very close to the construction of the functor $\mathcal{F}_{\mathrm{BVar}}$ in the proof of Theorem 3.1.1, so our exposition this time will be less detailed.

Step 1. *We can assume that X is regular and each B_i is a Cartier divisors. In particular, (X, B) is regular outside of Z.* First, we note that $\mathcal{F}_0(X, \emptyset, Z) : X' \dashrightarrow X$ is a desingularization of X (actually, it is $\mathcal{F}_{\mathrm{Var}}(X, Z)$ constructed in [Tem2, Th. 3.1.5]). Set $B' = f^\circ(B)$ and $Z' = f^*(Z)$). Then B' decomposes as $\overline{B} + \widetilde{B}$, where $\overline{B}$ is divisorial and any component of $\widetilde{B}$ is a component of X. Since any desingularization of $(X', \overline{B})$ is also a desingularization of (X', B') and hence of (X, B) we can safely replace (X, B, Z) with $(X', \overline{B}, Z')$ accomplishing the Step. In particular, the decomposition $B = B_{\mathrm{in}} + B_{\mathrm{out}}$ is now defined.

Now, we will define $\mathcal{F}(X, B, Z)$ as a composition of few T-supported blow up sequences with regular centers that will be denoted $f : X' \dashrightarrow X$ for ease of notation. We also use the notation $B' = f^\circ(B)$ and $Z' = f^*(Z)$. By bad loci we mean the closed sets T and $T' = (X', B')_{\mathrm{ssing}}$.

Step 2. *We can achieve that (X', B', Z') is in $\mathfrak{B}_1$, thus separating the outer boundary from the bad locus.*

Substep (a). *It is enough to construct $f : X' \dashrightarrow X$ such that $B'_{1,\mathrm{out}}$ is disjoint from T'.* Indeed, given such f we can apply the same argument to $(X', f^\circ(B), f^*(Z))$ to find a T'-supported blow up sequences $X'' \dashrightarrow X$ which separates separate another outer boundary component from the bad locus, etc.

Substep (b). *General plan of constructing f as in Substep (a).* We will only blow up the centers lying on the strict transforms of $\widetilde{X} := B_{1,\text{out}}$, and so f is the pushforward of the induced blow up sequence $\widetilde{f} : \widetilde{X}' \dashrightarrow \widetilde{X}$, where $\widetilde{X}' = f^!(\widetilde{X})$. Also, we will use the following notation: $\widetilde{Z} = Z|_{\widetilde{X}}$, $\widetilde{B} = (B - \{B_{1,\text{out}}\})|_{\widetilde{X}}$ and $\widetilde{B}' = \widetilde{f}^\circ(\widetilde{B})$, where $B - \{B_{1,\text{out}}\}$ is obtained from B by replacing B_1 with $B_{1,\text{in}} = B_1 - B_{1,\text{out}}$.

Substep (c). *We can achieve that the B-scheme $(\widetilde{X}', \widetilde{B}')$ is regular.* Since (X, B) is regular outside of Z, it follow that $(\widetilde{X}, \widetilde{B})$ is regular outside of $\widetilde{Z}$. Moreover, Z does not contain irreducible components of $\widetilde{X}$ and hence $\dim(\widetilde{Z}) = d - 1$ and $(\widetilde{X}, \widetilde{B}, \widetilde{Z})$ is an object of $\text{BSch}_{\text{small}}$. In particular, the desingularization $\widetilde{f} = \mathcal{F}_{\text{small}}(\widetilde{X}, \widetilde{B}, \widetilde{Z})$ is available for us by the induction assumption. Let $f : X' \dashrightarrow X$ be the pushforward of $\widetilde{f} : \widetilde{X}' \dashrightarrow \widetilde{X}$ with respect to the closed immersion $\widetilde{X} \hookrightarrow X$, then f is as required.

Substep (d). *We can achieve in addition that $\widetilde{X}'$ is a connected component of $f^\triangleright(B_1)$.* Since X' is regular and $\widetilde{X}'$ is its regular subscheme of codimension one, $\widetilde{X}'$ is a divisor. Clearly $\widetilde{X}' \hookrightarrow f^\triangleright(B_1)$ and hence $f^\triangleright(B_1) = \widetilde{X}' + Y'$ where Y' is a Z'-supported divisor. To accomplish the substep we should make $\widetilde{X}'$ disjoint from Y' and this can be done by successive applying Lemma 3.1.3 to components of $\widetilde{Y}' := Y'|_{\widetilde{X}'}$. The argument is a precise copy of Substep 2(e) in the proof of Theorem 3.1.1.

Substep (e). *The conditions from Substeps 4(c) and 4(d) imply that $\widetilde{X}'$ is disjoint from the bad locus T'. In particular, the condition of Substep 4(b) is achieved and this concludes the proof.* The argument copies Substep 2(f) in the proof of Theorem 3.1.1.

Step 3. *Resolution of B.* Since Step 2 is accomplished, we can simply consider $\mathcal{F}_1(X', B', Z') : X'' \dashrightarrow X'$ defined in Case 2 above, and the composition $g : X'' \dashrightarrow X$ is a desired desingularization of (X, B) that functorially depends only on the triple (X, B, Z). So, we set $\mathcal{F}_{\text{small}}(X, B, Z) = g$. $\square$

Remark 3.3.2. (i) We essentially used that $\widehat{\mathcal{F}}_{\text{BVar}}$ is strong (while in other Theorems of §3 the input desingularization functor does not have to be strong).

(ii) I do not know whether $\mathcal{F}_{\text{small}}$ is independent of Z, though probably it is.

3.4. General B-schemes.

3.4.1. *Unresolved locus.* Similarly to [Tem2, §4.1.1], when working on strong (resp. B-strong) desingularization of B-schemes by the *unresolved locus* f_{sing} of a B-blow up sequence $f : (X', B') \dashrightarrow (X, B)$ we mean the smallest closed subset $T \subset X$ such that f is a strong (resp. B-strong) desingularization over $X \setminus T$. Also, we say that f is a *desingularization up to codimension d* if $f_{\text{sing}} \subset X^{>d}$.

3.4.2. *Equicodimensional blow up sequences and filtration by codimension.* A B-blow up sequence $(X', B') \dashrightarrow (X, B)$ is *equicodimensional* if the blow up sequence $X' \dashrightarrow X$ is equicodimensional in the sense of [Tem2, §4.1.3]. There is a straightforward generalization of Lemma [Tem2, 4.1.3] to desingularization of B-scheme, which we leave to the reader. We will only need the obvious (and weaker) observation that if $\{\mathcal{F}^{\leq d}\}_{d \in \mathbf{N}}$ is a compatible family of functorial equicodimensional desingularizations up to codimension d then this family possesses a limit $\mathcal{F}$, which is a desingularization functor.

3.4.3. *Construction of $\mathcal{F}$.* Let BSch denote the category of all qe noetherian generically reduced B-schemes of characteristic zero with all exact regular morphisms.

Theorem 3.4.1. *Assume that there exists a desingularization $\mathcal{F}_{\text{small}}$ on $\mathrm{BSch}_{\text{small}}$. Then there exists a desingularization $\mathcal{F}$ on BSch. Moreover, if $\mathcal{F}_{\text{small}}$ is strong or B-strong, then $\mathcal{F}$ can be chosen to be strong or B-strong, respectively.*

Proof. We will show how to construct $\mathcal{F}$ on a B-scheme (X, B) from BSch, and it will be clear that all stages of the construction are functorial. Actually, we will build a compatible sequence of functors $\mathcal{F}^{\leq d}$ which provide an equicodimensional desingularization up to codimension d and such that the centers of $\mathcal{F}^{\leq d}(X, B)$ of X-codimension d sit over $T_{d-1} = \mathcal{F}^{\leq d-1}(X, B)_{\text{sing}}$. The construction will be done inductively, and we define $\mathcal{F}^{\leq 0}$ to be the trivial B-blow up because generically reduced B-scheme is semi-regular at all its maximal points.

In the sequel, we assume that the functors $\mathcal{F}^{\leq 0}, \ldots, \mathcal{F}^{\leq d-1}$ are already constructed, and our aim is to construct $\mathcal{F}^{\leq d}(X, B)$. The required B-blow up sequence will be obtained by modifying the B-blow up sequence $\mathcal{F}^{\leq d-1}(X, B)$. Let n be the length of this sequence. To simplify notation (and avoid double indexes), after each modification we will denote the obtained blow up sequence as $f : (X_m, B_m) \dashrightarrow (X_0, B_0) = (X, B)$. In particular, we start with $f = \mathcal{F}^{\leq d-1}(X, B)$ and $m = n$, and we will change f and m in the sequel. By our assumption, T_{d-1} is a closed subset of $X^{\geq d}$, hence it has finitely many points of codimension d. Let T denote the set of these points and $\overline{T}$ be its Zariski closure.

Step 1. *Construction of $\mathcal{F}^{\leq d}(X, B)$ in the particular case when $T = \overline{T}$ is closed.*

Note that in this case $\mathcal{F}^{\leq d-1}(X, B)$ is a desingularization over $X \setminus T$ and hence $\mathcal{F}^{\leq d}(X, B)$ will be a desingularization of the whole X. We will use the operation of inserting a blow up sequence into another blow up sequence, which is defined in [Tem2, Def. 4.2.2].

Extension 0. Provide T with the reduced scheme structure and extend f by inserting $\mathrm{Bl}_T(X) \to X$ as the first blow up. As an output we obtain a blow up sequence $\mathcal{F}_0^{\leq d}(X, B) : X'_m \dashrightarrow X'_0 \to X_0 = X$ of length $m + 1$ where the first center (the inserted one) is regular. Set $f = \mathcal{F}_0^{\leq d}(X, B)$ and increase m by one after this step. As an output we achieve that $T \times_X X_i$ is a Cartier divisor in X_i for $i > 0$.

Extensions $1, \ldots, n$. The last n centers of $\mathcal{F}_0^{\leq d}(X, B)$ does not have to be suitable for a B-strong (resp. strong) desingularization. So, we will use n successive extensions to make the centers B-permissible (reps. regular) in the case of B-strong (resp. strong) desingularization. If the desingularization is not strong, one should just skip these extensions and go directly to extension $n + 1$. Let us describe the i-th extension with $1 \leq i \leq n$. It obtains as an input a blow up sequence $f = \mathcal{F}_{i-1}^{\leq d}(X, B)$ in which only the last $n - i$ centers can be non-permissible (resp. non-regular) and outputs a blow up sequence $\mathcal{F}_i^{\leq d}(X, B)$ with only $n - i - 1$ bad blow ups in the end. By our assumption, $(X_{i+1}, B_{i+1}) \to (X_i, B_i)$ is the first blow up of f whose center W can be non-permissible (resp. non-regular). The latter happens if and only if $T_W := (W, B_i|_W)_{\text{ssing}}$ (resp. $T_W := W_{\text{sing}}$) is not empty. Obviously, the bad locus T_W sits over T and hence $T_W \subset W \cap (X_i)^d \subset W^{d-1}$. In particular, $\mathcal{F}^{\leq d-1}(W, B_i|_W)$ is a B-strong (resp. strong) desingularization $h : (W', \widetilde{B}') \dashrightarrow (W, B_i|_W)$ which is T_W-supported and hence T-supported. By [Tem2, Lemma 4.2.1], the pushforward $H : X'_i \dashrightarrow X_i$ of h with respect to the closed immersion $W \hookrightarrow X_i$ is a blow up

sequence with the same centers. Moreover, H is B_i-permissible in the B-strong case by Lemma 2.3.3.

Let now $f' : X'_m \dashrightarrow X'_i \dashrightarrow X_i \dashrightarrow X_0$ be obtained from f by inserting $H : X'_i \dashrightarrow X_i$ instead of $X_{i+1} \to X_i$. By [Tem2, Lemma 4.2.3] the center of $X'_{i+1} \to X'_i$ is the strict transform of W, hence it is W'. Since $(W', \widetilde{B}')$ is semi-regular (resp. W' is regular) and $\widetilde{B}' = B'_i|_{W'}$, only the last $i-1$ blow ups of f' can have non-permissible (resp. non-regular) centers. So, we can set $\mathcal{F}^{\leq d}_i(X, B) = f'$ and replace the old f with f'.

Extension $n{+}1$. At this stage we already have a blow up sequence $f = \mathcal{F}^{\leq d}_{\widetilde{n}}(X, B)$ such that all its centers are as required. The last problem we have to resolve is that $(X_m, B_m)_{\mathrm{ssing}}$ does not have to be empty. However, we at least know that the bad locus is T-supported and hence is contained in the Cartier divisor $D = T \times_X X_m$, which is a disjoint union of varieties. So, the triple (X_m, B_m, D) is an object of $\mathrm{BSch}_{\mathrm{small}}$ and hence (X_m, B_m) can be desingularized by $\mathcal{F}_{\mathrm{small}}(X_m, B_m, D)$.

Step 2. Construction of $\mathcal{F}^{\leq d}(X, B)$ in general. Set $X_T = \sqcup_{x \in T} \mathrm{Spec}(\mathcal{O}_{X,x})$ and $B_T = g^*(B)$, where $g : X_T \to X$ is the projection, and note that $f_T = \mathcal{F}^{\leq d}(X_T, B_T)$ was defined in Step 1. For each $x \in T$ let f_x denote the restriction of f_T onto $X_x = \mathrm{Spec}(\mathcal{O}_{X,x})$ with all trivial blow ups inherited from f_T. By functoriality f_x is a trivial extension of $\mathcal{F}^{\leq d}(X_x, B|_{X_x})$ (which is also defined by Step 1). Choose an open neighborhood $U \hookrightarrow X$ of $X^{\leq d}$ such that the closures $\overline{x} \in U$ of distinct points $x \in T$ are pairwise disjoint, and define $g_x : U_x \dashrightarrow U_m$ as the pushforward of f_x with respect to the pro-open immersion $X_x \hookrightarrow U$. Since each g_x is $\overline{x}$-supported, [Tem2, Lemma 4.2.4] implies that we can merge all g_x's into a single blow up sequence $g : U' \dashrightarrow U$. Finally, we define $f : X' \dashrightarrow X$ to be the pushforward of g with respect to the open immersion $U \hookrightarrow X$. It follows that f is obtained from $\mathcal{F}^{\leq d-1}(X, B)$ by inserting few equicodimensional $\overline{T}$-supported blow ups. In particular, f is a trivial extension of $\mathcal{F}^{\leq d-1}(X, B)$ over $X \setminus \overline{T}$, and f coincides with f_x over X_x for each $x \in T$. Thus, f desingularizes (X, B) over $X^{\leq d}$ and so we can set $f = \mathcal{F}^{\leq d}(X, B)$.

Functoriality of the construction is established by checking that all intermediate steps were functorial. This is straightforward and is done in the same way as in the proof of [Tem2, 1.2.1], so we omit the details. $\square$

3.5. **Proof of the main results.**

3.5.1. *Theorems 1.1.4.* Applying Theorems 3.1.1, 3.2.3, 3.3.1 and 3.4.1 one after another we construct a strong desingularization functor $\mathcal{F}$, thus proving Theorem 1.1.4.

3.5.2. *Theorem 1.1.6.* Before constructing $\mathcal{F}_{\mathrm{emb}}(X, E, Z)$ we should extend (and modify) the functor $\mathcal{F}$ slightly.

Step 0. *Construct a functor $\widetilde{\mathcal{F}}$ which is an analog of $\mathcal{F}$ but applies to all noetherian qe B-schemes of characteristic zero.* Given such a B-scheme (X, B), consider the reduction $\widetilde{X}$ of X with the pushforward $f : X' \dashrightarrow X$ of $\mathcal{F}(\widetilde{X}, \emptyset) : \widetilde{X}' \dashrightarrow \widetilde{X}$ and set $B' = f^\circ(B)$. Note that $\widetilde{X}'$ is the reduction of X', so we can blow up all components of $\widetilde{X}'$ which underlie generically non-reduced components of X'. This gives a B-blow up $(X'', B'') \to (X', B')$ with generically reduced source. Finally, if $(X''', B''') \dashrightarrow (X'', B'')$ is the blow up sequence $\mathcal{F}(X'', B'')$, then the composition

$(X''', B''') \dashrightarrow (X, B)$ is a required strong desingularization of (X, B), which we denote $\widetilde{\mathcal{F}}(X, B)$.

Note that $\widetilde{\mathcal{F}}$ provides a stupid way to desingularize a generically non-reduced B-scheme, which we mentioned in Remark 2.4.1(iii), and not the clever way mentioned in Remark 1.1.5(ii). Now, we will construct $\mathcal{F}_{\mathrm{emb}}(X, E, Z)$ by composing two T-supported blow up sequences with regular centers, where the bad locus T consists of all points where Z has not simple normal crossings with E (e.g. Z is not regular). We will denote the intermediate sequence $f : X' \dashrightarrow X$ and we set, in addition, $Z' = f^!(Z)$ and $E' = f^\circ(E)$.

Step 1. *We can achieve that Z' is regular and has simple normal crossings with E'.* Consider the B-blow up sequence $\widetilde{f} : (Z', B') \dashrightarrow (Z, B)$, where $B = E|_Z$ and $\widetilde{f} = \widetilde{\mathcal{F}}(Z, B)$, and let $f : (X', E') \dashrightarrow (X, E)$ be its pushforward. In particular, the centers of f are as required, and by Lemma 2.2.8 $B' = E'|_{Z'}$. Thus, $E'_{Z'}$ is semi-regular and hence Z' has simple normal crossings with E'.

Step 2. *Making the boundary snc.* Since $\widetilde{f}$, and hence f, does not have to be permissible, it can happen that E' is not snc. However, Z' has simple normal crossings with E', and so the bad locus of (X', E') is disjoint from Z'. Thus, $\mathcal{F}(X', E') : (X'', E'') \dashrightarrow (X', E')$ does not change anything in a neighborhood of Z' and makes the boundary snc (because (X'', E'') is semi-regular and E'' is divisorial). Also, this B-blow up sequence is T-supported because E' does not have to be snc only at the points $x' \in X'$ where f is not an isomorphism. So, the composition $g : (X'', E'') \dashrightarrow (X, E)$ satisfies all conditions of the Theorem and we can set $\mathcal{F}_{\mathrm{emb}}(X, E, Z) = g$.

3.5.3. *Other Theorems.* We saw in the Introduction that other main results follow from the above two. Namely, Theorem 1.1.2 is a particular case of Theorem 1.1.4, and we also showed in the Introduction how to deduce Theorems 1.1.7 and 1.1.8.

Appendix A

A.1. **Motivation for introducing B-schemes.** In this appendix we discuss where B-schemes come from, and why their usage seems to be very natural. In particular, we will discuss their relation to the classical desingularization approach.

A.1.1. *Embedded desingularization and the boundary.* It is now a common knowledge (at least since the great work [Hir] of Hironaka) that in the embedded desingularization one should give a special treatment to the exceptional divisor accumulated during the blow up sequence, and it is also common to call this divisor the boundary (because in many situation it behaves as a kind of boundary). Thus, for the sake of mastering an inductive desingularization procedure one should consider triples (X, E, Z) even if one starts with (and uses in applications) the case when E is empty.

Also, it is now a standard observation that although the support of E is an snc divisor, one should provide E with the finer structure of splitting to regular components and (at least to some extent) with ordering of these components (see also §A.1.3). Very naturally, both tasks are accomplished by the history of the desingularization process, and, after adding the history function, E becomes an snc boundary in our sense. Note, in addition, that the rule of forming the new boundary from the old one is the complete transform.

A.1.2. *Non-embedded desingularization and the boundary.* A common approach to building a non-embedded desingularization of a scheme Z is to embed it into a regular ambient scheme X and to apply embedded desingularization to $(X, \emptyset, Z)$. (For example, this gives a non-strong desingularization of equidimensional varieties of characteristic zero in many works, including [Wł] and [Kol].) As an output one gets an admissible blow up sequence $f : X' \dashrightarrow X$ such that $\widetilde{f} : Z' = f^!(Z) \dashrightarrow Z$ is a desingularization and Z' has simple normal crossings with E' (the boundary of f). In particular, not only Z is regular but also the exceptional divisor $E'|_{Z'}$ of $\widetilde{f}$ is snc. Moreover, if we add an arbitrary initial snc boundary E such that $D := E|_Z$ is a divisor and resolve (X, E, Z) by $f : X' \dashrightarrow X$, then its restriction $\widetilde{f} : Z' \dashrightarrow Z$ not only outputs regular Z' but also makes $D \times_Z Z'$ to a strictly monomial divisor whose reduction has simple normal crossings with the exceptional divisor.

To summarize the above paragraph, the embedded desingularization makes more than just to desingularize the embedded scheme Z – it desingularizes the embedded pair $(Z, E|_Z)$ (and this is non-trivial even when $E|_Z = \emptyset$). This observation indicates that even for non-embedded desingularization one naturally (though usually implicitly) deals with boundaries and the natural problem one solves (even without planning to) is to desingularize a pair (Z, B) where $B = E|_Z$ is a boundary. Since E is a set of divisors, B should at least be a set of divisors rather than a single divisor. Moreover, it is natural to restrict E on any closed subscheme without excluding degenerate cases and this directly leads to our definition of boundaries.

A.1.3. *On strict desingularization from* [Tem1]. In addition to the general motivation discussed in §A.1.2 let us discuss a concrete example that illustrates why the *naive boundary* defined as a divisor does not work as fine as the boundary in our sense. Actually, desingularization with naive boundaries was studied in [Tem1]. For a pair (Z, D), where Z is a variety of characteristic zero and D is a divisor (or a closed subscheme), one can combine embedded and non-embedded desingularization to find a blow up sequence $f : Z' \dashrightarrow Z$ such that Z' is regular, $D' = f^*(D)$ is strictly monomial and f only modifies $Z_{\mathrm{sing}} \cup D_{\mathrm{sing}}$. This fact was used in [Tem1] to prove a similar result for qe schemes over **Q**. It seemed to me natural to expect that there should exist f such that D' is (strictly) monomial and f only modifies the union of X_{sing} and the not (strictly) monomial locus of D, and such f was called *(semi) strict desingularization* in [Tem1].

Remark A.1.1. (i) Strict desingularization does not exists even for algebraic surfaces, as one can see in the classical example of Whitney umbrella, which is discussed below in Example A.1.2. In particular, there is no semi-strict functorial desingularization, as follows (iii) below.

(ii) Existence of strict desingularization of varieties was incorrectly proved in [Tem1, 2.2.11]. The mistake in that proof was in claim (i) and it is due to the fact that the number of formal brunches through a point of D is not Zariski semi-continuous, unlike the number of irreducible components. This should be corrected by replacing strict desingularization and formal branches with semi-strict desingularization and irreducible components in the formulation/proof of [Tem1, 2.2.11]. Actually this was in the original argument I heard from Bierstone-Milman! The correction does not affect anything else in the paper.

(iii) Any functorial semi-strict desingularization must be strict because monomial and strictly monomial loci are indistinguishable in the étale topology. In particular,

the above semi-strict desingularization is not fully functorial. Indeed, it obviously follows from the construction that it is only functorial with respect to the morphisms that preserve the number of irreducible components through any point (and missing this subtlety lead me to the mistake).

(iv) Clearly, the above trouble is taken care of when one works with boundaries $D = \{D_1, \ldots, D_n\}$, their snc loci and exact regular morphisms (recall that by Remark 2.1.4 the snc locus of D is contained in the snc locus of $[D]$ but can be strictly smaller). Furthermore, the k-multiple locus $D(k)$, whose points lie exactly on k boundary components, is functorial in (Z, D) with respect to exact regular morphisms. Thus, the proof of [Tem1, 2.2.11] applies to the B-variety (Z, D) and provides a strong functorial desingularization of B-varieties, thus giving another proof of Theorem 2.4.2. Moreover, this proof has an advantage that it does not use the order and hence applies to unordered boundaries as well.

Example A.1.2. Take the Whitney umbrella $D \subset Z = \mathbf{A}_k^3 = \mathrm{Spec}(k[x, y, z])$, which is given by $x^2 = zy^2$. A classical observation (which I learned on a lecture of H. Hauser) is as follows. The only non-monomial point of D is the origin (though there are many points that are not strictly monomial), hence any strict desingularization which only blows up smooth centers must blow up this point as the first step. A simple computation shows that the new pair (Z', D') has a unique singular point in the exceptional locus and this point is of the same type $x_1^2 = zy_1^2$. So, we must blow up that point, and ad infinitum. Moreover, J. Kollar proved by a more involved argument that (Z, D) does not admit any strict desingularization (at least when $\mathrm{char}(k) = 0$).

A.1.4. *B-permissibility.* As we discussed in §A.1.1, the blow up sequence of the ambient B-scheme (X, E) is usually E-permissible. Although usually one uses embedded desingularization to construct the non-embedded one, sometimes one goes in the opposite direction. For example, this is (actually) the case with the strong embedded desingularizations from [BM1, 12.2] or [BM2, 6.8]. In this case (an initial part of) the blow up sequence of X is obtained by blowing up centers on the strict transforms of Z. So, the blow up sequence $f : X' \dashrightarrow X$ is the pushforward of a blow up sequence $\widetilde{f} : Z' \dashrightarrow Z$. If one also wants to encode in $\widetilde{f}$ the information that f is E-permissible, then one naturally obtains the general definition of boundaries and permissibility. Namely, Lemma 2.3.3 says that f is E-permissible if and only if $\widetilde{f}$ is $(Z, E|_Z)$-permissible. We used this way of reasoning (i.e. $\widetilde{f}$ is permissible $\Rightarrow$ f is permissible) in the proof of Theorem 3.4.1.

A.1.5. *B-schemes.* In order to discuss desingularization of pairs (Z, B) it looks natural to link them into a single object, and the fact that such a pair can be interpreted as a log-scheme of a special type gives a strong indication that this definition makes sense. We will discuss below two situations where the use of B-schemes seems to be very natural.

Remark A.1.3. It is an interesting question whether more general log-schemes can be useful for desingularization theory.

A.1.6. *Redundancy of blow ups.* There are two possibilities of what to call a blow up of schemes. In [Tem1] by a blow up one means a morphism $X' \to X$ that is isomorphic to a blow up (with some center), while in [Tem2] and in this paper the center of a blow up is a part of the data, and so blow ups are enriched morphisms.

The latter is crucial in order to have strict and principal transforms. There are also various examples (obvious and not) of different blow up sequences that produce the same morphism but induce different tranforms, see [Kol, 3.33]. However, if we consider a blow up of schemes $f : X' \to X$ as a B-blow up of B-schemes $(X', E') \to (X, \emptyset)$, then this redundancy disappears (and similarly for the blow up sequences from [Kol]). The only small redundancy with B-blow ups was described in Remark 2.2.7(i), and even that could be avoided by formal using of the history function for ordering (i.e. by ordering the components by the natural numbers so that empty components are allowed).

A.1.7. *Transforms.* Principal (weak or controllable) transform of closed subschemes (or ideals) is commonly used in embedded desingularization. The idea is to split off from the ideal some multiples of the exceptional divisors until nothing is left. In general, there is no morphism $(X', f^{\triangleright}(B)) \to (X, B)$, and the complete transform of B is actually the minimal natural increment of $f^{\triangleright}(B)$ such that there is a natural morphism of B-schemes $(X', f^{\circ}(B)) \to (X, B)$. In particular, $|f^{\circ}(B)| = |f^!(|B|)| \cup |E_f|$ and so the complete transform keeps (at least set-theoretical) information about the old boundary and the boundary of f.

A.1.8. *B-strong desingularization conjecture.* The language of B-schemes gives a natural formulation of a desingularization conjecture which encodes both embedded and non-embedded desingularizations (but not the principalization).

Conjecture A.1.4. *For any qe generically reduced B-scheme (X, B) of characteristic zero there exists a B-strong desingularization $\mathcal{F}(X, B)$ which is functorial in all exact regular morphisms.*

Remark A.1.5. (i) I hope that this holds for all qe schemes, but I prefer to formulate the Conjecture only in the characteristic zero case, where it should be provable by today's technique.

(ii) One can formulate a generalization for non-reduced B-schemes. This would require to introduce/recall some terminology (normal flatness, etc.), so we prefer not to deal with this in this paper. Similarly, I conjecture that this B-strong desingularization can be obtained by only blowing up at the maximal Hilbert-Samuel strata.

(iii) Currently, the conjecture is open even for B-varieties, and the main obstacle to proving it is that one should first define a presentation of the Hilbert-Samuel function (in the sense of [BMT, §6]) on a B-variety (X, B). For a B-variety $(X, \emptyset)$ we can simply use the presentation on X, and then the Conjecture holds true by Theorem 2.4.2.

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